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Turgay Bayraktar and Emel Karaca

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Complex analysis and geometry, Algebraic geometry / *Analyse et géométrie complexes, Géométrie algébrique*

An Exponential Rarefaction Result for Sub-Gaussian Real Algebraic Maximal Curves

Un résultat de raréfaction exponentielle pour les courbes algébriques maximales réelles sous-Gaussiennes

Turgay Bayraktar [*,a] **and Emel Karaca** [b]

[a] Faculty of Engineering and Natural Sciences, Sabancı University, İstanbul, 34956 Turkey

[b] Polatlı Faculty of Science and Arts, Ankara Hacı Bayram Veli University, Ankara, 06900 Turkey

E-mails: tbayraktar@sabanciuniv.edu, emel.karaca@hbv.edu.tr

Abstract. We prove that maximal real algebraic curves associated with sub-Gaussian random real holomorphic sections of a smoothly curved ample line bundle are exponentially rare. This generalizes the result of Gayet and Welschinger [13] proved in the Gaussian case for positively curved real holomorphic line bundles.

Résumé. Nous démontrons que les courbes algébriques réelles maximales associées aux sections holomorphes réelles sous-Gaussiennes d'un faisceau de lignes amples à courbure lisse sont exponentiellement raréfiées. Cela généralise le résultat de Gayet et Welschinger [13] ont prouvé dans le cas Gaussien des faisceaux de lignes holomorphes réels à courbure positive.

1. Introduction

Let X be a projective manifold of dimension m and $L \to X$ be an ample line bundle. Assume that the real locus $\mathbf{R}X$ is non-empty and $\pi : (L, c_L) \to (X, c_X)$ is a real holomorphic line bundle. The latter means that $\pi : L \to X$ is a holomorphic line bundle and the anti-holomorphic involutions c_X and c_L satisfy $c_X \circ \pi = \pi \circ c_L$. We denote the vector space of global holomorphic sections of the tensor power $L^{\otimes n}$ by $H^0(X, L^{\otimes n})$, its dimension is finite and will be denoted by d_n (see eg. [14]). We also let

$$\mathbf{R}H^0\left(X, L^{\otimes n}\right) := \left\{s \in H^0\left(X, L^{\otimes n}\right) : c_L \circ s = s \circ c_X\right\}$$

be the space of real sections whose real dimension is d_n. The discriminant locus (respectively, its real part) will be denoted by $\Delta_n \subset H^0(X, L^n)$ (respectively, $\mathbf{R}\Delta_n$), it is the set of global holomorphic

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[*] Corresponding author

sections with singular zero locus. For $s \in H^0(X, L^n)$ we let $Z_s = s^{-1}(0)$ (respectively, $\mathbf{R}Z_s$) be the complex (respectively real) zero locus of s. Note that for $s \in H^0(X, L^n) \setminus \Delta_n$ the hypersurface Z_s is a complex manifold. For $s \in \mathbf{R}H^0(X, L^{\otimes n}) \setminus \mathbf{R}\Delta_n$ we denote the number of connected components of the real zero locus by $b_0(\mathbf{R}Z_s)$.

Let h be a singular Hermitian metric on L. Recall that for a holomorphic frame e_L of L on an open set $U \subset X$, we have $|e_L|_h = e^{-\varphi}$ for some $\varphi \in L^1_{loc}(U)$ (see e.g. [7]). The function φ is called *local weight function* of the metric h with respect to the frame e_L. Moreover, the curvature current is given by $c_1(L, h)_{|U} = dd^c\varphi$ where $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$. We say that a singular Hermitian metric is positive (respectively of class $\mathscr{C}^\alpha$) if the local weight functions are psh (respectively $\mathscr{C}^\alpha$) functions. Given a $\mathscr{C}^2$ Hermitian metric $h = e^{-\varphi}$ on L one can define an *extremal metric* $h_e := e^{-\varphi_e}$ by taking the upper envelope of all local weight functions of positive metrics dominated by φ (see [5, § 3]). More precisely, for the holomorphic frame e_L on the open set U we have

$$\varphi_e := \sup \left\{\psi \text{ is a psh local weight} : \psi \le \varphi \text{ on } U\right\}. \tag{1}$$

It turns out that $h_e = e^{-\varphi_e}$ defines a singular Hermitian metric on L such that its local weights are $\mathscr{C}^{1,1}$ psh functions. We denote its curvature current by $dd^c\varphi_e$; it is a globally well-defined positive closed $(1,1)$ current on X. By [5], the *equilibrium measure* is defined by

$$\mu_{\varphi_e} := \left(dd^c\varphi_e\right)^m / m!$$

supported on the compact set

$$S := \overline{X_h^+ \cap D}$$

where $D := \{x \in X : \varphi(x) = \varphi_e(x)\}$ and the set

$$X_h^+ := \left\{x \in X : dd^c\varphi(x) > 0\right\}.$$

Note that the set X_h^+ depends only on the metric h but not on the local weight φ.

The geometric data given above allow us to define a Hermitian inner product on the vector space of *global holomorphic sections* $H^0(X, L^{\otimes n})$ via

$$\langle s_1, s_2 \rangle := \int_X \langle s_1(x), s_2(x) \rangle_{h^{\otimes n}} \, dV \tag{2}$$

where dV is a fixed volume form on X. The restriction of (2) to $\mathbf{R}H^0(X, L^{\otimes n})$ induces an Euclidean inner product. We also denote the induced norm by $\| \cdot \|_n$.

Let $\{S_j^n\}_{j=1}^{d_n}$ be an orthonormal basis for $\mathbf{R}H^0(X, L^{\otimes n})$ with respect to the inner product (2). We consider *sub-Gaussian random holomorphic sections* that are of the form

$$S_n := \sum_{j=1}^{d_n} c_j^n S_j^n$$

where c_j^n are i.i.d. real sub-Gaussian random variables of mean zero and unit variance on some fixed probability space (see § 3.1 for the definition and standart examples of sub-Gaussian random variables). This induces a d_n-fold product probability measure $Prob_n$ on the vector space $\mathbf{R}H^0(X, L^{\otimes n})$. We remark that unlike the Gaussian case, the probability structure in this more general setting depends on the choice of the orthonormal basis. Throughout this note, we assume that the probability distribution of c_j is absolutely continuous with respect to Lebesgue measure and hence, with $Prob_n$ probability one the real zero loci $\mathbf{R}Z_{s_n}$ is non-singular. We also consider the product probability space

$$\prod_{n=1}^{\infty} \left(\mathbf{R}H^0\left(X, L^{\otimes n}\right), Prob_n\right) \tag{3}$$

whose elements are sequences of random real holomorphic sections. We refer the reader to the manuscripts [2, 3] for examples of local probabilistic models.

When X is a projective surface, it follows from Harnack-Klein inequality that the number of real connected components of a non-singular real algebraic curve $b_0(\mathbf{R}Z_s)$ is bounded by $g(Z_s) + 1$. Here, $g(Z_s)$ denotes the genus of the Riemann surface Z_s and it is given by the adjunction formula $g(Z_s) = \frac{1}{2}(n^2 L^2 - n c_1(X)L + 2)$, where $c_1(X)$ is the first Chern class of X. For $n \in \mathbb{N} \setminus \{0\}$ and $a > 0$ we consider the set

$$M_n^a := \left\{ s \in \mathbf{R}H^0\left(X, L^n\right) \setminus \mathbf{R}\Delta_n : b_0\left(\mathbf{R}Z_s\right) \geq g(Z_s) + 1 - an \right\}. \tag{4}$$

The set M_n^a is always non-empty for sufficiently large n (see e.g. [13]). Gayet and Welschinger [13, Theorem 1] proved that when (L, h) is positively curved (i.e. the curvature form $c_1(L, h) > 0$) and the random coefficients c_j^n are i.i.d. standard Gaussians, the probability of the set M_n^a decreases exponentially fast. More recently, Diatta and Lerario [9] (see also [1]) generalized this result to higher dimensions by using a different approach: namely, with high probability for a Gaussian random real algebraic section one can approximate the real zero locus without changing its topology by real zero loci of a section associated with a lower tensor power.

In this note, we generalize [13, Theorem 1] in three directions: we allow merely smooth Hermitian metric on the holomorphic line bundle L and let the random coefficients c_j^n be i.i.d. continuous sub-Gaussian random variables with mean zero and unit variance. Moreover, the exponential decay rate is far better. More precisely, we prove the following result:

Theorem 1. *Let X be a real projective surface and $L \to X$ be a real holomorphic ample line bundle endowed with a real $\mathscr{C}^2$ metric $h = e^{-\varphi}$. Assume that X_h^+ is non-empty. Then for each $a > 0$ there exists constants $C > 0$ and $n_0 \in \mathbb{N}$ such that*

$$Prob_n\left(M_n^a\right) \leq C n^2 e^{-n^2} \tag{5}$$

for $n \geq n_0$.

In the proof of Theorem 1, we rely on the observation by Gayet and Welschinger [13] that for real holomorphic sections contained in M_n^a if the corresponding normalized (complex) current of integrations along Z_{s_n} converge to a positive closed $(1, 1)$ current T then T is weakly laminar outside the real locus $\mathbf{R}X$ (Proposition 13). In the current geometric setting, the distribution of the complex zeros is almost surely determined by the extremal current $dd^c \varphi_e$ (Theorem 12). Moreover, $dd^c \varphi_e$ is nowhere weakly laminar in the bulk (Proposition 6). In order to estimate the probability of the exceptional set M_n^a we use mass asymptotics (Theorem 11) together with Hanson-Wright Inequality (Theorem 9) which leads to a sharper exponential decay rate than the one obtained in [13, Theorem 1].

2. Preliminaries

2.1. *Laminar currents*

We begin with some definitions and basic results on laminar currents that will be useful in the sequel. Since the definitions are local we fix a domain Ω in $\mathbb{C}^2$ and let $\mathscr{D}^{p,q}(\Omega)$ be the set of (smooth) test forms with compact support in Ω. The dual space $\mathscr{D}_{p,q}(\Omega)$ is the set of (p, q) currents. A $(1, 1)$ current T is called *closed* if $\langle T, d\phi \rangle = 0$ for every test 1-form. We say that T is *positive* if $\langle T, \phi \rangle \geq 0$ for every positive $(1, 1)$ test form. We refer the reader to the manuscript [8] for a detailed account of positive closed currents.

For a positive closed $(1, 1)$ current T we denote its (closed) support by $supp(T)$ and let $\|T\|$ be its trace measure and $M(T)$ be its mass norm. For example, if C is a (possibly singular) holomorphic curve in Ω then the pairing

$$\langle [C], \phi \rangle := \int_{C_{reg}} \phi \text{ for } \phi \in \mathscr{D}^{1,1}(\Omega) \tag{6}$$

defines a positive closed $(1,1)$ current in Ω and called *current of integration* along C.

Definition 2. *A $(1,1)$ current T is called uniformly laminar if for each $x \in supp(T)$ there exists an open set U containing x and biholomorhic to the unit polydisc $\mathbb{D}^2$ and a measurable family of holomorphic functions $f_a : \mathbb{D} \to \mathbb{D}$ together with a measure $d\lambda$ on the set of parameters $\mathscr{A} := \{f_a(0)\}$ such that the graphs $\Gamma_a = \{(x, f_a(y)) : y \in \mathbb{D}\}$ are pairwise disjoint and we have*

$$\langle T, \phi \rangle = \int_{\mathscr{A}} \left(\int_{\Gamma_a} \phi \right) d\lambda(a)$$

for each $\phi \in \mathscr{D}^{1,1}(U)$.

A uniformly laminar current is necessarily positive and closed. Clearly, a current of integration along a smooth algebraic curve of the form (6) is uniformly laminar. Demailly [6] proved that every positive closed current supported on a $\mathscr{C}^1$ Levi-flat hypersurface in Ω is uniformly laminar.

For two positive closed currents T_1, T_2 we may write $T_j = dd^c u_j$ for some psh functions on Ω. We say that the exterior product $T_1 \wedge T_2$ is *admissible* if $u_1 \in L^1_{loc}(\|T_2\|)$. Note that this condition is independent of the choice of the potential u_1 for T_1. In this case, the current $u_1 T_2$ has locally bounded mass and the exterior product

$$T_1 \wedge T_2 := dd^c(u_1 T_2)$$

is a positive measure with finite mass. We also remark that the exterior product is symmetric that is $u_1 \in L^1_{loc}(\|T_2\|)$ if and only if $u_2 \in L^1_{loc}(\|T_1\|)$ and

$$T_1 \wedge T_2 = T_2 \wedge T_1.$$

Let S_j be positive closed currents satisfying $S_j \leq T_j$ (that is $T_j - S_j$ is positive) for $j = 1, 2$. It is well-known that if $T_1 \wedge T_2$ is admissible then $S_1 \wedge S_2$ is also admissible and we have

$$S_1 \wedge S_2 \leq T_1 \wedge T_2.$$

We will need the the following result in the sequel (see [11, § 6]):

Proposition 3. *Let $T = dd^c u$ for some psh fonction u on Ω for some $u \in L^\infty_{loc}(\Omega)$. Assume that T is uniformly laminar. Then $T \wedge T = 0$ on Ω.*

Weakly laminar currents were introduced by Bedford-Lyubich-Smillie [4] as a natural generalization of uniformly laminar currents. They arise naturally in pluri-potential theory and complex dynamics. We refer the reader to the papers [4, 10] and references therein for more details.

Definition 4. *A positive closed $(1,1)$ current T in Ω is called weakly laminar if for each $\epsilon > 0$ there exists a uniformly laminar current current T_ϵ on a subdomains $\Omega_\epsilon \subset \Omega$ such that $T_\epsilon \leq T$ and $\|T_\epsilon\|(\partial\Omega_\epsilon) = 0$ satisfying $M((T - T_\epsilon)|_{\Omega_\epsilon}) < \epsilon$.*

Equivalently (see [4]), T is weakly laminar if there exists a measurable family $(\mathscr{A}, \lambda)$ of embedded holomorphic discs $D_a \subset \Omega$ such that for every pair (a, b) the overlap $D_a \cap D_b$ is either empty or an open set in the disc topology and we have

$$T = \int_{\mathscr{A}} [D_a] d\lambda(a). \tag{7}$$

The following are examples illustrating the difference between weakly laminar currents and uniformly laminar ones (see [6] and [4, § 5] for more details):

Example 5. Let (x, y) denote coordinates on $\mathbb{C}^2$ and

$$u_1(x, y) = \frac{1}{2} \log^+ \left(|x|^2 + |y|^2 \right)$$

$$u_2(x, y) = \log^+ \left(\max\{|x|, |y|\} \right)$$

where $\log^+ = \max(\log, 0)$. The currents $dd^c u_1$ and $dd^c u_2$ are extremal and weakly laminar but they are not uniformly laminar currents.

Gayet and Welschinger [13, Lemma 1] observed that a Kähler form does not have any laminar structure. We generalize this result to our geometric setting by proving that the extremal current associated with a smooth Hermitian metric does not have any laminar structure in the bulk:

Proposition 6. *Let φ_e be the extremal metric defined by* (1). *Then $dd^c\varphi_e$ is nowhere weakly laminar in the set $Int(D \cap X_h^+)$.*

Proof. Assume on the contrary that $T := dd^c\varphi_e$ is weakly laminar near some point $z \in \Omega :=$ $Int(D \cap X_h^+)$. Then we can find uniformly laminar currents $T_n \le T$ on an increasing sequence of subdomains such that $z \in \Omega_n \subset \Omega$ and

$$M((T - T_n)|_{\Omega_n}) < \frac{1}{n}. \tag{8}$$

Since $T_n \to T$ weakly, we may find negative psh potentials u_n (resp. u) for T_n (resp. T) such that $u_n \to u$ in L^1_{loc}. Since $T_n \le T$ the function $u - u_n$ is psh and we may assume that $u \le u_n \le 0$. Thus, $u_n T_n$ has locally bounded mass. Moreover, by [12, Proposition 3.2] we have

$$T_n \wedge T_n \to T \wedge T = \mu_{\varphi_e}$$

weakly. On the other hand, by [5, Theorem 3.4] there exists $\delta > 0$ such that

$$\mu_{\varphi_e}(\Omega_n) = \int_{\Omega_n} \det\left(dd^c\varphi\right) dV > \delta$$

which contradicts Proposition 3.   □

The following result gives sufficient conditions for weak laminarity:

Theorem 7 (de Thélin). *Let $(C_n)_{n \in \mathbf{N}}$ be a sequence of smooth holomorphic curves in the unit polydisc in $\mathbb{C}^2$ such that $g(C_n) = O(A(C_n))$, where $g(C_n)$ stands for the genus of C_n and $A(C_n)$ for its area. If $\frac{1}{A(C_n)}[C_n]$ converges to a positive closed $(1,1)$ current T as n grows to infinity then T is weakly laminar.*

3. Quantum Ergodicity for Random Holomorphic Sections

3.1. *Sub-Gaussian random variables*

We recall some basic properties of Sub-Gaussian random variables. Assume that $(\Omega, \mathscr{F}, \tau)$ is a probability space. A real valued random variable $X : \Omega \to \mathbb{R}$ is called *Sub-Gaussian* with parameter $b > 0$ if the moment generating function (MGF) of X is dominated by MGF of a normalized Gaussian, that is there exists $b > 0$ such that

$$\mathbb{E}\left[e^{tX}\right] \le e^{\frac{b^2 t^2}{2}}$$

for all $t \in \mathbb{R}$. The classical examples of Sub-Gaussian random variables are Standard Gaussian $N(0,1)$, Bernoulli random variables $\mathbb{P}[X = \pm 1] = \frac{1}{2}$, and uniform distribution on $[-1,1]$. We have the following characterizations for sub-Gaussian random variables:

Proposition 8 ([18, Lemma 5.5]). *Let X be a centered real random variable (i.e. $\mathbb{E}[X] = 0$). Then the following equations are equivalent:*

(1) $\exists\, b > 0$ *such that* $\mathbb{E}[e^{tX}] \le e^{\frac{b^2 t^2}{2}}$ *for all $t \in \mathbb{R}$.*
(2) $\exists\, c > 0$ *such that* $\mathbb{P}[|X| > \alpha] \le 2e^{-c\alpha^2}$ *for every $\alpha > 0$.*
(3) $\exists\, K > 0$ *such that* $(\mathbb{E}(|X|^p)^{\frac{1}{p}}) \le K\sqrt{p}$ *for all $p \ge 1$.*

(4) $\exists\, \kappa > 0$ *such that* $\mathbb{E}[e^{\frac{X^2}{\kappa^2}}] \leq 2$.

The last property is known as ψ_2 *condition.* Recall that for a centered random variable X the *Orlicz norm* is defined by

$$\|X\|_{\psi_2} := \sup_{p \geq 1} p^{-1/2} \left(\mathbb{E}|X|^p\right)^{1/p}. \tag{9}$$

In particular, X is Sub-Gaussian if and only if $\|X\|_{\psi_2}$ is finite.

3.2. *Hanson–Wright Inequality*

Let X_j be independent Sub-Gaussian random variables and $\kappa_j := \|X_j\|_{\psi_2}$. The joint probability distribution of $X := (X_1, \ldots, X_N)$ is denoted by $\mathbb{P}$. We also let $A = [A_{ij}]$ be a square matrix with real entries. We denote its operator norm by

$$\|A\| := \max_{\|v\|_2 \leq 1} \|Av\|.$$

where $\|.\|_2$ is the Euclidean norm. We also denote the Hilbert-Schmidt norm by

$$\|A\|_{HS} := \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}} = \left[Tr\left(AA^T\right)\right]^{\frac{1}{2}}. \tag{10}$$

In [15], Hanson–Wright proved a concentration inequality for the values of a quadratic form

$$X \to X^T A X$$

acting on a random vector X. The following version is due to Rudelson-Vershynin [16]:

Theorem 9 (Hanson–Wright Inequality). *Assume that A is a $N \times N$ square matrix and $X = (X_1, \ldots, X_N) \in \mathbb{R}^N$ is a random vector with components X_j being independent Sub-Gaussian variables such that*

$$\left\|X_j\right\|_{\psi_2} \leq K$$

for $j = 1, 2, \ldots, N$. Then for each $t \geq 0$

$$\mathbb{P}\left[\left|X^T A X - \mathbb{E}\left[X^T A X\right]\right| > t\right] \leq 2\exp\left(-c\min\left\{\frac{t^2}{K^4 \|A\|_{HS}^2}, \frac{t}{K^2 \|A\|}\right\}\right)$$

where $c > 0$ is an absolute constant which does not depend on t.

3.3. *Bergman Kernel Asymptotics*

Let $X, L, e^{-\varphi}$ be as in the introduction and $\dim_{\mathbb{C}} X = m$. Recall that the *Bergman kernel* for the L^2-space $(H^0(X, L^{\otimes n}), \langle,\rangle)$ is the integral kernel of the orthogonal projection from L^2 space of global sections with values in $L^{\otimes n}$ onto $H^0(X, L^{\otimes n})$. Note that the Bergman kernel satisfies the reproducing property

$$s(x) = \int_X \left\langle s(x), K_n(x, y)\right\rangle_{h^{\otimes n}} dV(y)$$

for $s \in H^0(X, L^{\otimes n})$. For any orthonormal basis $\{S_j^n\}$ for $H^0(X, L^{\otimes n})$ the Bergman kernel can be obtained as a smooth section

$$K_n(x, y) = \sum_{j=1}^{d_n} S_j^n(x) \otimes \overline{S_j^n(y)}$$

of the line bundle $L^{\otimes n} \boxtimes (L^{\otimes n})^*$ over $X \times X$. Note that this representation is independent of the choice of the orthonormal basis S_j^n. The point-wise norm of the restriction of $K_n(x, y)$ to the diagonal is called the *Bergman function*; it is given by

$$k_n(x) := |K_n(x, x)| = \sum_{j=1}^{d_n} \left| S_j^n \right|_{h^{\otimes n}}^2.$$

We remark that $k_n(x)$ has the dimensional density property:

$$\int_X k_n(x) dV = \dim\left(H^0\left(X, L^{\otimes n} \right) \right) = d_n$$

for $n \in \mathbb{N}$.

Theorem 10 ([5, Theorem 1.3]). *Let X be a real projective manifold and $L \to X$ be a real holomorphic line bundle endowed with a real $\mathscr{C}^2$ metric h. Then the following weak convergence of measures holds:*

$$n^{-m} k_n dV \to \mu_{\varphi_e},$$

Moreover,

$$n^{-m} k_n(x) \to \mathbf{1}_{D \cap X_h^+} det\left(dd^c \varphi \right)(x)$$

for almost every x in X, where X_h^+ is the set where $dd^c \varphi > 0$.

3.4. *Mass Asymptotics*

Let $X, L, h = e^{-\varphi}$ be as in the introduction. We consider the scalar L^2- product and the induced norm on the vector space of global holomorphic sections $H^0(X, L^{\otimes n})$ given by

$$\|s\|_n^2 := \int_X |s|_{h^{\otimes n}}^2 dV$$

where dV denotes the probability volume form induced by $w = c_1(L, h)$. When (L, h) is positively curved, that is $dd^c \varphi > 0$, Shiffman and Zelditch [17] proved that for a sequence $(s_n)_n \in H^0(X, L^{\otimes n})$ of global holomorphic sections if their *masses*

$$\frac{1}{d_n} |s|_{h^{\otimes n}}^2 dV \to dV$$

in the weak-star topology of measures on X then the normalized current of integration $\frac{1}{n}[Z_{s_n}]$ also converge to the curvature form $c_1(L, h)$ in the sense of currents. More recently, Zelditch [20] obtained a generalization of this result in the setting of partially positive metrics on positive line bundles. However, the proof of [20, Theorem 1.2] had a gap and it was fixed in [3]:

Theorem 11. *[3, 20] Let X be a real projective manifold and $L \to X$ be a real holomorphic line bundle endowed with a real $\mathscr{C}^2$ metric h. Let $(s_n)_n$ be a sequence of global holomorphic sections. Assume that the masses*

$$\frac{1}{d_n} |s_n(z)|_{h^{\otimes n}}^2 dV \to d\mu_{\varphi_e} \ as \ n \to \infty \tag{11}$$

in the weak-star sense on $S = \overline{X_h^+ \cap D}$. Then the normalized currents of integration

$$\frac{1}{n}\left[Z_{s_n} \right] \to dd^c \varphi_e$$

in the sense of currents.

We also have a random version of Theorem 11 in setting of sub-Gaussian holomorphic sections:

Theorem 12 ([3, Theorem 5.1]). *Let X be a real projective manifold and $L \to X$ be a real holomorphic line bundle endowed with a real $\mathscr{C}^2$ metric h. Then for almost every sequence $(s_n)_n$ in $\prod_{n=1}^{\infty} (\mathbf{R}H^0(X, L^{\otimes n}), Prob_n)$ the masses*

$$\frac{1}{d_n} |s_n(z)|_{h^{\otimes n}}^2 \, dV \to d\mu_{\varphi_e} \tag{12}$$

in the weak-star sense on $S = \overline{X_h^+ \cap D}$.

4. Proof of Theorem 1

The following result was obtained in [13]. We provide a proof for the sake of completeness.

Proposition 13. *Let C_n be a sequence of smooth algebraic curves in X of degree ρ_n satisfying*

$$g(C_n) + 1 - b_0(\mathbf{R}C_n) = O(\rho_n).$$

Assume that the normalized currents of integration $\frac{1}{\rho_n}[C_n]$ converge weakly to a positive closed $(1,1)$ current T on X. Then T is weakly laminar on $X \setminus \mathbf{R}X$.

Proof. By Theorem 7, it is enough to prove that for each ball $B \subset X \setminus \mathbf{R}X$ we have

$$g(C_n \cap B) = O(A_n)$$

where $g(C_n \cap B)$ is the genus of the closed surface obtained by capping a disc at each boundary component of $C_n \cap B$ and $A_n = Area(C_n \cap B)$. To this end, first we observe that $g(C_n \setminus \mathbf{R}C_n) = O(\rho_n)$. Indeed, its Euler characteristic satisfies

$$\chi(C_n \setminus \mathbf{R}C_n) = \chi(C_n) - \chi(\mathbf{R}C_n) = 2 - 2g(C_n).$$

The genus of the Riemann surface $g(C_n \setminus \mathbf{R}C_n)$ is given by the formula

$$\chi(C_n \setminus \mathbf{R}C_n) = 2b_0(C_n \setminus \mathbf{R}C_n) - 2g(C_n \setminus \mathbf{R}C_n) - r(C_n \setminus \mathbf{R}C_n)$$

where $r(C_n \setminus \mathbf{R}C_n) = 2b_0(\mathbf{R}C_n)$ is the number of ends of Riemann surface $C_n \setminus \mathbf{R}C_n$. By the argument on [19, page 66] we have $b_0(C_n \setminus \mathbf{R}C_n) \leq 2$ which implies that

$$\begin{aligned} g(C_n \setminus \mathbf{R}C_n) &= b_0(C_n \setminus \mathbf{R}C_n) - \frac{1}{2}\chi(C_n \setminus \mathbf{R}C_n) - \frac{1}{2}r(C_n \setminus \mathbf{R}C_n) \\ &\leq 1 + g(C_n) - b_0(\mathbf{R}C_n) \\ &= O(\rho_n). \end{aligned}$$

Now, the Euler characteristic

$$\chi(C_n \cap B) = 2b_0(C_n \cap B) - 2g(C_n \cap B) - r(C_n \cap B).$$

Since the genus of the orientable surface can only increase under connected sums, we see that $g(C_n \cap B) \leq g(C_n \setminus \mathbf{R}C_n) = O(\rho_n)$. Finally, since $\frac{1}{\rho_n}[C_n] \to T$ we see that for sufficiently large n we have $A(C_n \cap B) \geq c\rho_n$ for some $c > 0$ depending on T and B. Thus, the assertion follows from Theorem 7. $\qquad\square$

Proof of Theorem 1. First, we observe that

$$Prob_n \left[s_n \in \mathbf{R}H^0\left(X, L^{\otimes n}\right) : \|s_n\|_n > d_n \right] \leq d_n \exp(-d_n).$$

Indeed, for $s_n := \sum_{j=1}^{d_n} c_j^n S_j^n$ we have

$$\begin{aligned} Prob_n [\|s_n\|_n > d_n] &= Prob_n \left[\sum_{j=1}^{d_n} \left| c_j^n \right|^2 > d_n^2 \right] \\ &\leq Prob_n \left[\left| c_j^n \right|^2 > d_n \quad \text{for some } j \right] \\ &\leq 2d_n \exp(-d_n) = O\left(d_n \exp(-d_n)\right). \end{aligned} \tag{13}$$

Let $R_n := M_n^a \cap \{\|s_n\|_n \le d_n\}$ and $\mathscr{M}(X)$ be the set of Borel measures of total mass at most one. We also let

$$\Omega_n := \left\{ v_{s_n} : \text{ for some } s_n \in R_n \right\} \subset \mathscr{M}(X)$$

where $v_{s_n} := \frac{1}{d_n} |s_n|_{h^{\otimes n}}^2 dV$ denotes the mass of s_n. Note that in order to prove (5), by (13), it is enough to show that

$$Prob_n(R_n) = O\left(\exp(-d_n)\right).$$

To this end, let us consider the compact set $\Omega := \overline{\cup \Omega_n} \subset \mathscr{M}(X)$ where the closure is taken with respect to the weak-star topology. Note that by Proposition 6 and Theorem 11 we have $\mu_{\phi_e} \notin \Omega$. Hence, by compactness there exists an $\epsilon > 0$ and finitely many test functions ϕ_j on $S_\phi \backslash \mathbf{R}X$ such that for each $v_{s_n} \in \Omega$ we have

$$\left| \frac{1}{d_n} X_n^{\phi_j}(s_n) - \int_X \phi_j d\mu_{\varphi_e} \right| > \epsilon \tag{14}$$

where

$$X_n^{\phi_j}(s_n) = \int_X \phi_j |s_n(z)|_{h^{\otimes n}}^2 \, dV.$$

Next, by identifying $s_n \in \mathbf{R}H^0(X, L^n)$ with a vector in $\mathbb{R}^{d_n}$ and using linear algebra one can find an $d_n \times d_n$ symmetric matrix $A_n^{\phi_j}$ such that

$$X_n^{\phi_j}(s_n) = \left\langle A_n^{\phi_j} s_n, s_n \right\rangle =: s_n^T A_n^{\phi_j} s_n.$$

where $\langle, \rangle$ is the standard inner product on $\mathbb{R}^{d_n}$. Observe that $A_n^{\phi_j}$ is nothing but the matrix representing the Toeplitz operator $T_n^{\phi_j} := \pi_n M_{\phi_j}$, where π is the Bergman projection and M_{ϕ_j} is the multiplication operator induced by ϕ_j.

By [3, Proposition 3.8] we have

$$\mathbb{E}\left[X_n^{\phi_j} \right] = \int_X \phi_j k_n(z) dV = Tr\left(T_n^{\phi_j} \right).$$

Then by Theorem 9 there exists an absolute constant $C > 0$ such that

$$Prob_n\left[\left| X_n^{\phi_j}(s_n) - \mathbb{E}\left[X_n^{\phi_j} \right] \right| > t \right] =$$

$$\mathbb{P}_n\left[\left| s_n^T A_n^{\phi_j} s_n - \mathbb{E}\left[s_n^T A_n^{\phi_j} s_n \right] \right| > t \right] \le 2 \exp\left(-C \min\left\{ \frac{t^2}{K^4 \left\| A_n^{\phi_j} \right\|_{HS}^2}, \frac{t}{K^2 \left\| A_n^{\phi_j} \right\|} \right\} \right), \tag{15}$$

where $K := \|c_j^n\|_{\psi_2} \ge \frac{1}{\sqrt{2}}$. Note that the operator norm satisfies $\|A_n^{\phi_j}\| \le \sup_{z \in X} |\phi_j(z)|$. Moreover, since $A_n^{\phi_j}$ is symmetric by (10) and [3, Proposition 3.3] we have

$$\left\| A_n^{\phi_j} \right\|_{HS}^2 = Tr\left(\left(T_n^{\phi_j} \right)^2 \right) = O(d_n).$$

Furthermore, by Theorem 10 and using $d_n = O(n^2)$ we have

$$\left| \int_X \phi_j d\mu_{\varphi_e} - \frac{1}{n^2} \mathbb{E}\left[X_n^{\phi_j} \right] \right| < \frac{\epsilon}{2}$$

for all j and sufficiently large n. Then by applying triangle inequality first and letting $t = \epsilon d_n$ in (15) we obtain

$$Prob_n\left[\left| \frac{1}{d_n} X_n^{\phi_j}(s_n) - \int_X \phi_j d\mu_{\varphi_e} \right| > \epsilon \right] \le Prob_n\left[\left| \frac{1}{d_n} X_n^{\phi_j}(s_n) - \frac{1}{n^2} \mathbb{E}\left[X_n^{\phi_j} \right] \right| > \epsilon/2 \right]$$

$$= O\left(\exp\left(-n^2\right)\right) \tag{16}$$

Thus, combining (13) and (16) we deduce that

$$Prob_n\left[M_n^a \right] = O\left(n^2 \exp\left(-n^2\right)\right). \qquad \square$$

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